ON THE SOLUTION OF EQUATIONS.

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THE irrational roots of equations of all degrees may be determined by series. Thus, if

$$a = x + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + gx^7 + \&c. \qquad \text{(G)}$$

we may find

$$\begin{aligned} x = a - ba^2 &+ (2b^2 - c)a^3 - (5b^3 - 5bc + d)a^4 + (14b^4 - 21b^2c + 6bd + 3c^2 - e)a^5 \\ &- (42b^5 - 84b^3c + 28b^2d + 28bc^2 - 7be - 7cd + f)a^6 \\ &+ (132b^6 - 330b^4c + 120b^3d + 180b^2c^2 - 36b^2e - 72bcd + 8bf + 8ce - 12c^3 \\ &\qquad + 4d^2 - g)a^7 - \&c., \qquad (R) \end{aligned}$$

by assuming $x = a + Aa^2 + Ba^3 + Ca^4 + Da^5 + Ea^6 + Fa^7 + \&c.$, in (G).

Now let us take the following equations, viz.;

$$a = x + 0x^2 + 0x^3 + 0x^4 + \&c., \text{ an eq'n of the 1st degree.}$$

$$a = x + bx^2 + 0x^3 + 0x^4 + \text{ " " " " " 2nd " .}$$

$$\left.\begin{array}{l} a = x + bx^2 + cx^3 + 0x^4 + \text{ "} \\ a = x + 0x^2 + cx^3 + 0x^4 + \text{ "} \end{array}\right\} \text{eq'ns " " 3rd " .}$$

$$\left.\begin{array}{l} a = x + bx^2 + cx^3 + dx^4 + 0x^5 + \&c., \\ a = x + 0x^2 + cx^3 + dx^4 + 0x^5 + \&c., \\ a = x + 0x^2 + 0x^3 + dx^4 + 0x^5 + \&c., \end{array}\right\} \text{eq's of 4th degree.}$$

$$\left.\begin{array}{l} a = x + bx^2 + cx^3 + dx^4 + ex^5 + 0x^6 + \&c., \\ a = x + bx^2 + cx^3 + 0x^4 + ex^5 + 0x^6 + \&c., \\ a = x + 0x^2 + cx^3 + 0x^4 + ex^5 + 0x^6 + \&c., \\ a = x + 0x^2 + 0x^3 + 0x^4 + ex^5 + 0x^6 + \&c., \end{array}\right\} \text{eq'ns of 5th deg., \&c.}$$

From which it appears that equation (G) is a complete expression for an equation of any degree, and that its series may be made to vanish at any term, so as to give any degree, by giving zero values to such of the coefficients b, c, d, &c. as the case may require.

The series in (R), however, will not end after the first degree, but its terms may be greatly contracted, in many cases, by cancelling all of those having zero coefficients. Thus, if in both (G) and (R) we make $c = 0$, $d = 0$, $e = 0$, &c., we get,

$$a = x + bx^2, \qquad (G')$$

$$x = a - ba^2 + 2b^2a^3 - 5b^3a^4 + 14b^4a^5 - 42b^5a^6 + 132a^7 - \&c. \qquad (R')$$

1. Given $x^2 + 10x = 1$, or $\frac{1}{10} = x + \frac{1}{10}x^2$. Put $a = \frac{1}{10}$ and $b = \frac{1}{10}$ in (R'), then $x = \frac{1}{10} - \frac{1}{10^3} + \frac{2}{10^5} - \frac{5}{10^7} + \&c. = .0990195$

2. $x^2 + 10x = 5$, or $\frac{1}{2} = x + \frac{1}{10}x^2$. Here $a = \frac{1}{2}$, $b = \frac{1}{10}$; therefore

$$x = \frac{1}{2} - \frac{1}{10.2^2} + \frac{2}{10^2.2^3} - \frac{5}{10^3.2^4} + \frac{14}{10^4.2^5} - \frac{42}{10^5.2^6} + \frac{132}{10^6.2^7} = .477225$$

3. $15x^2 - 50x + 7 = 0$, or $\frac{7}{50} = x - \frac{3}{10}x^2$. Here $a = \frac{7}{50}$, $b = -\frac{3}{10}$;

$$\therefore x = \frac{7}{50} + \frac{3.7^2}{10.50^2} + \frac{2\,3^2.7^3}{10^2.50^3} + \frac{5.3^3.7^4}{10^3.50^4} + \&c.$$

4. Given $x^2+x=1$. Here $a=1$ and $b=1$; $\therefore x = 1-2+5-14$, &c.

5. $x^2+x = 100$. $a = 100, b = 1$; $x = 100-2.100^2+5.100^3-$&c.

From Examples 1, 2 and 3, we see that the values of a and b in formula (R') when applied should be small fractions or the series will not converge rapidly, but may diverge as in Ex's 4 and 5. Hence direct application of the formula would only be practically useful in comparatively few cases.

There is, however, an indirect method, as presented in my Math. Key, which makes the formula applicable to all cases. Thus, in the equation $x^2+x=100$, we may soon find by trial that x lies between 9 and 10, then put $x = \frac{19}{2} + y$, and we have $(\frac{19}{2}+y)^2+(\frac{19}{2}+y) = 100$, or $\frac{1}{4.20} = y + \frac{1}{20}y^2$;

$$\therefore x = \frac{19}{2}+y = \frac{19}{2}+\frac{1}{4.20}-\frac{1}{4^2.20^3}+\frac{2}{4^3.20^5}-\text{\&c.} = 9.51249.$$

6. Given, $x^2+6x = 8$, to find x. As the value is near 1, let $x = 1+y$. Then the eq. may be written $\frac{1}{8}=y+\frac{1}{8}y^2$, and $x=1+y = 1+\frac{1}{8}-\frac{1}{8^3}+\frac{2}{8^5}$ &c.

7. Required the square root of 2, or the value of x in the eq'n $x^2 = 2$.

Let $x = 1.4+y$, as 1.4 is near the value of x. Then $(1.4+y)^2 = 1.96+2.8y+y^2 = 2$, or $\frac{1}{70} = y+\frac{5}{14}y^2$. Developing y by (R') as before, we find

$$x = \frac{14}{10}+\frac{1}{70}-\frac{5}{14.70^2}+\frac{2.5^2}{14^2.70^3}-\text{\&c.}$$

In all the examples here given other series might be found that would converge much faster, by taking a nearer value of x before transforming.

In the solution of cubic equations, we may consider the values of all the letters zero, except a, b, c and x, in both (G) and (R), and by then cancel'g all zero terms we have $a = x+bx^2+cx^3$; and for its root,

$$x = a-ba^2+(2b^2-c)a^3-(5b^3-5bc)a^4+(14b^4-21b^2c+3c^2)a^5-\text{\&c.}; \quad (R'')$$

or, when $b = 0$, $\qquad a = x + cx^3$, and

$$x = a-ca^3+3c^2a^5-12c^3a^7+\text{\&c.} \qquad (R''')$$

8. Given $x^3-3x^2+10x = 1$, or $\frac{1}{10} = x-\frac{3}{10}x^2+\frac{1}{10}x^3$. Let $a = \frac{1}{10}$, $b = \frac{3}{10}$ and $c = \frac{1}{10}$ in (R''); then $x = \frac{1}{10}+\frac{3}{10^3}+\frac{8}{10^5}-\frac{15}{10^7}-$&c. $= .1031$, nearly.

9. Given, $x^4-3x^2+75x = 10000$, to find one root of the equation.

As upon trial x is found nearly equal 10, let $x = 9.9+y$, and the equation becomes, $-.0139676 = y+.15y^2+$&c. Developing by (R') we get

$$x = 9.9+y = 9.9-.0139676-.0000293-.0000001-\text{\&c.} = 9.8860027.$$

When the fractions of the new equation have large terms it is best to reduce to decimals, as in the last case.

10. Given, $x^5-30x^4+340^3-1800x^2+4384x = 3841$, to find all the r'ts.

As the several values of x are near to 2, 4, 6, 8 and 10, respectively; $\therefore$ substitute successively $x_1 = 2+y$, $x_2 = 4+y$, $x_3 = 6+y$, $x_4 = 8+y$ and $x_5 = 10+y$. The new equations will then be

$$\frac{1}{4.96} = y - \frac{100}{96}y^2 + \frac{35}{96}y^3 - \frac{5}{96}y^4 + \frac{1}{4.96}y^5,$$
$$-\frac{1}{96} = y - \frac{40}{96}y^2 - \frac{20}{96}y^3 + \frac{10}{96}y^4 - \frac{1}{96}y^5,$$
$$+\frac{1}{64} = y + 0\ y^2 - \frac{20}{64}y^3 + 0\ y^4 + \frac{1}{64}y^5,$$
$$-\frac{1}{96} = y + \frac{40}{96}y^2 - \frac{20}{96}y^3 - \frac{10}{96}y^4 - \frac{1}{96}y^5,$$
$$\frac{1}{4.96} = y + \frac{100}{96}y^2 + \frac{35}{96}y^3 + \frac{5}{96}y^4 + \frac{1}{4.96}y^5.$$

If we now develop each of these equations by (R) and add to the results the corresponding integers, 2, 4, 6, 8 and 10, we shall have the five roots of the equation. Thus,

$$x_1 = 2 + \frac{1}{4.96} + \frac{100}{4^2.96^3} + \frac{16640}{4^3.96^5} + \ldots = 2.002611,$$
$$x_2 = 4 - \frac{1}{96} + \frac{40}{96^3} - \frac{5120}{96^5} + \ldots = 3.989591,$$
$$x_3 = 6 + \frac{1}{64} + \frac{20}{64^4} + \frac{1136}{64^7} + \ldots = 6.015626,$$
$$x_4 = 8 - \frac{1}{96} - \frac{40}{96^3} - \frac{5120}{96^5} - \ldots = 7.989568,$$
$$x_5 = 10 + \frac{1}{4.96} - \frac{100}{4^2.96^3} + \frac{16640}{4^3.96^3} - \ldots = 10.002597.$$

In like manner all irrational roots may be found by the rule expressed by (R) which may therefore be called the Root Theorem.

GEOMETRICAL DETERMINATION OF THE AREA OF THE PARABOLA.

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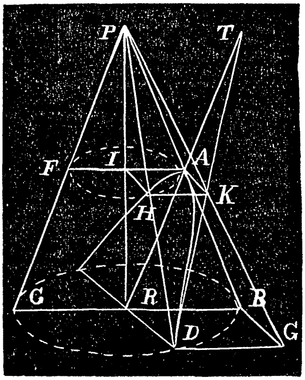

LET AR represent the principal axis of a parabola, so placed as to form one of the equal sides of an isosceles triangle ABR, with the base BR, equal to the corresponding ordinate of the given parabola, and at R erect a perpendicular RP to intersect AB produced in P.

Suppose the cone $PBDC$ completed by revolving PBR about PR; then will CPB, also, be isosceles, and similar to RAB, hence the line AR is parallel to PC, and AB is half of PB.

Along the line AR pass a plane with its cutt'g